If we take the 2nd solution of θ $(s\theta < 0)$

$\rightarrow \phi = \operatorname{atan2}(-u_{13}, -u_{23})$

$\rightarrow \psi = \operatorname{atan2}(u_{31}, -u_{32})$

If $u_{13} = u_{23} = 0 \Rightarrow u_{33} = \pm 1$ and

$u_{31} = u_{32} = 0$

If $u_{33} = 1 \Rightarrow c\theta = 1$ and $s\theta = 0 \Rightarrow \theta = 0$

$$\begin{bmatrix} c(\phi+\psi) & -s(\phi+\psi) & 0 \\ s(\phi+\psi) & c(\phi+\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & 0 \\ u_{21} & u_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{s(\phi+\psi)}{c(\phi+\psi)} = \frac{u_{21}}{u_{11}} = \tan(\phi+\psi) \Rightarrow \phi+\psi = \operatorname{atan2}(u_{11}, u_{21}) = \operatorname{atan2}(u_{11}, -u_{12})$$

<u>$\theta=0$ is a singular configuration</u> because we have infinity of solutions for the ψ and ϕ since we have just the sum $(\phi+\psi)$.

If $u_{33} = -1$ then $c\theta = -1$ and $s\theta = 0 \Rightarrow \boxed{\theta = \pi}$

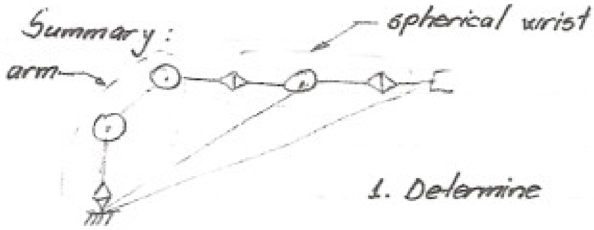
In this case we have

$$\begin{bmatrix} -c(\phi-\psi) & -s(\phi-\psi) & 0 \\ s(\phi-\psi) & c(\phi-\psi) & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & 0 \\ u_{21} & u_{22} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\frac{-u_{12}}{-u_{11}} = \frac{s(\phi-\psi)}{c(\phi-\psi)} = \tan(\phi-\psi) \Rightarrow \phi-\psi = \operatorname{atan2}(-u_{11}, -u_{12}) = \operatorname{atan2}(-u_{21}, -u_{22}).$$

<u>$\theta=\pi$ is a singular configuration</u> because only the difference $(\phi-\psi)$ is determined and hence we have infinity of solutions for ϕ and ψ.

Summary:



$${}^0_6H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \text{ given}$$

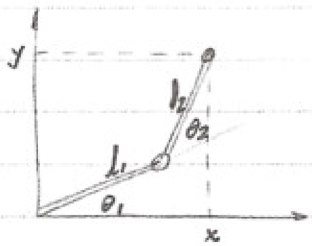
1. Determine $P_c = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} d_x - d_6 r_{13} \\ d_y - d_6 r_{23} \\ d_z - d_6 r_{33} \end{bmatrix}$

2. Solve for the first 3 joint variables (for the arm)
3. Having the 1st three joint variables get ${}^0_3R$ (forward kinematics)
4. Obtain ${}^3_6R = {}^0_3R^T R$

5 ${}^3_6R = U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix}$ and use the procedure solving the inverse kinematics for the spherical wrist with $\phi = \theta_4$, $\theta = \theta_5$, $\psi = \theta_6$.

<u>Algebraic Method</u> (J. Craig page 131)

<u>Velocity Kinematics</u>



$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

$$\dot{x} = -l_1 \dot{\theta}_1 \sin\theta_1 - l_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2)$$
$$\dot{y} = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 (\dot{\theta}_2 + \dot{\theta}_2) \cos(\theta_1 + \theta_2)$$

$$\underset{\substack{\text{linear}\\\text{velocity of}\\\text{end-effector}}}{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}} = \underbrace{\begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}}_{\substack{J(\theta_1, \theta_2)\\\text{manipulator Jacobian.}}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J(\theta_1, \theta_2) \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

Inverse kinematics: given $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ find $\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = J^{-1}(\theta_1, \theta_2) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$

$\det(J(\theta_1, \theta_2)) \neq 0.$

The values of θ_1 & θ_2 that provide $\det(J(\theta_1, \theta_2)) = 0$ define the singular configurations.

$$\theta_2 = 0 \Rightarrow J = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin\theta_1 & -l_2 \sin\theta_1 \\ (l_1 + l_2)\cos\theta_1 & l_2 \cos\theta_1 \end{bmatrix}$$

$$\det J = -l_2 (l_1 + l_2) \sin\theta_1 \cos\theta_1 + l_2 (l_1 + l_2) \cos\theta_1 \sin\theta_1 = 0$$

$\theta_2 = 0 \Rightarrow$ singular configuration

$$\underbrace{\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}}_{6} = \underbrace{J}_{6\times n} \underbrace{\begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}}_{n}$$

n-DOF manipulator

$J(q_1 \cdots q_n)$